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Inverse buckling problem for inhomogeneous columns ☆

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Abstract

This study is motivated by our recent study (Elishakoff, I., Rollot, O., 1999. New closed-form solutions for buckling of a variable stiffness column by mathematica, Journal of Sound and Vibration 224, 172–182), which presented new closed-form solutions for buckling of columns with variable stiffness. The column was simply supported at both ends. In the present study, we rederive, in other means, the same solution, and, moreover, obtain new solutions for two other sets of the boundary conditions by posing an inverse buckling problem. It is shown that the buckling load is dependent upon a single stiffness coefficient. By suitable choice of this parameter, the buckling load can be made arbitrarily large. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Buckling; Inverse problems; Bernoulli–Euler columns

1. Introduction

The exact solutions for buckling load of uniform beams are treated in almost any textbook on mechanics of solids. Exact solutions for non-uniform columns is the subject of several works (see, e.g. the textbook by Timoshenko and Gere (1961)). These solutions are derived in terms of Bessel or Lommel functions, or some other, elementary or transcendental functions. As far as the closed-form solutions are concerned, the results are much more restricted. All existing closed-form solutions are apparently listed, along with new solutions, in the recent study by Elishakoff and Rollot (1999).

This study is devoted to obtaining additional closed-form solutions, which are posed as inverse problems. Namely, the formulation of the problem is as follows: Find the polynomial distribution of the Young modulus $E(x)$ of an inhomogeneous column of the uniform cross-section with specified boundary conditions, so that the buckling mode will be a pre-selected polynomial function. It turns out that this seemingly simple formulation allows one to derive new closed-form solutions. The obtained solutions appear to be of much importance once the technologies are available to construct columns with given variation of modulus of elasticity, any pre-selected buckling load can then be achieved for the appropriate design of the structure.

☆ Dedicated to 80th Birth Anniversary of Professor Stephen Harry Crandall.

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2. Formulation of the problem

The differential equation that governs the buckling of the column under axial load P, reads

$$\frac{d^2}{dx^2}\left[D(x)\frac{d^2w}{dx^2}\right]+P\frac{d^2w}{dx^2}=0, \tag{1}$$

where $w(x)$ is the transverse displacement, $D(x)=E(x)I(x)$ is the stiffness, $E(x)$ is the modulus of elasticity, $I(x)$ is the moment of inertia, x is the axial coordinate. We consider three sets of boundary conditions. For the column that is simply supported at its both ends, the simplest polynomial that satisfies the boundary conditions at $x=0$ and $x=L$, with L is the length of the column, reads

$$\psi(\xi)=\xi-2\xi^3+\xi^4, \qquad \xi=x/L. \tag{2}$$

We pose the following question: Is there a column with polynomial variation of $E(\xi)$ that possesses the function in Eq. (2) as its fundamental buckling mode? Indeed, if the sought solution exists, it corresponds to the fundamental buckling load since $\psi(\xi)$ in Eq. (2) does not have internal nodes. This problem differs from the *direct* buckling problem, which pre-supposes the knowledge of the stiffness D and requires the determination of the mode $w(x)$ and the buckling load P. Here, we are looking for the *cause*, i.e., the distribution of the stiffness while knowing the *effect*, by the buckling mode.

We are looking for the stiffness $D(\xi)$ represented as follows:

$$D(\xi)=b_0+b_1\xi+b_2\xi^2, \tag{3}$$

where b_0, b_1 and b_2 are sought constants. The inverse problems may have no solution, multiple solutions or unique solution. It turns out that in the case under study, a unique solution exists for reconstructing the column, that possesses the function in Eq. (2) as its buckling mode, once a single parameter, namely b_2 is specified. We now consider the different sets of boundary conditions.

3. Column simply supported at its both ends

Utilization of the non-dimensional axial coordinate ξ, defined in Eq. (2) reduces the governing Eq. (1) to

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2\psi}{d\xi^2}\right]+PL^2\frac{d^2\psi}{d\xi^2}=0. \tag{4}$$

With the buckling mode postulated in Eq. (2), we have for the term in Eq. (4),

$$PL^2\frac{d^2\psi}{d\xi^2}=PL^2\left(-12\xi+12\xi^2\right), \tag{5}$$

whereas the first differential expression in Eq. (4) reads

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2\psi}{d\xi^2}\right]=-12\left[2(b_1-b_0)+6\xi(b_2-b_1)-12b_2\xi^2\right]. \tag{6}$$

The sum of expressions on the right-hand sides in Eqs. (5) and (6) must vanish, due to Eq. (4). Since the above sum must equal to zero identically, for any value of ξ, we get following expressions:

$$2(b_1-b_0)=0, \tag{7}$$

$$-72(b_2-b_1)-12PL^2=0, \tag{8}$$

$$144b_2 + 12PL^2 = 0. \tag{9}$$

Solution of Eq. (9) yields

$$P = -12\frac{b_2}{L^2}. \tag{10}$$

In order that the load P remains compressive, it must be positive. We conclude, therefore, that the coefficient b_2 must be negative. Eqs. (7) and (8) lead, with Eq. (10) taken into account, to

$$b_0 = b_1 = -b_2. \tag{11}$$

Thus, the stiffness is defined up to the coefficient b_2:

$$D(\xi) = \left(-1 - \xi + \xi^2\right)b_2. \tag{12}$$

We have already established that b_2 must take a negative value. Hence, Eq. (12) can be rewritten as

$$D(\xi) = \left(1 + \xi - \xi^2\right)|b_2|. \tag{13}$$

This function is in agreement with the physical realizability condition, namely, with the requirement of positivity of the the function $D(\xi)$ in the interval [0;1]. We thus have found the function $D(\xi)$ that corresponds to the postulated buckling mode in Eq. (2). As is seen, the solution of the posed problem is a unique one in the class of polynomially varying stiffnesses once b_2 is specified. Note that Eq. (10) coincides with Eq. (56) in the paper by Elishakoff and Rollot (1999). It pertains to the column that is simply supported at its both ends. We will show that Eq. (10) is valid for two other sets of boundary conditions. Note that using the Bubnov–Galerkin method to the above nonuniform column, with the comparison function $\psi(\xi) = \sin(\pi\xi)$, the exact mode shape of the associated uniform column yields the buckling load $(3 + 7\pi^2)/(b_2/6L^2)$, constituting 0.12% error in comparison with Eq. (10).

4. Column clamped at its both ends

The boundary conditions,

$$w = 0, \quad \frac{\mathrm{d}w}{\mathrm{d}\xi} = 0, \quad \text{at } \xi = 0, \quad \xi = 1, \tag{14}$$

are satisfied for the following polynomial function:

$$\psi(\xi) = \xi^2 - 2\xi^3 + \xi^4. \tag{15}$$

We are interested in establishing if this polynomial function may serve as a buckling shape of any inhomogeneous column. The expression for $PL^2\psi''$ reads, with primes denoting differentiation with respect to ξ:

$$PL^2\psi'' = PL^2\left(2 - 12\xi + 12\xi^2\right), \tag{16}$$

whereas the expression for $(D\psi'')''$ is

$$\left(D\psi''\right)'' = 2(2b_2 - 12b_1 + 12b_0) + 6(-12b_2 + 12b_1)\xi + 144b_2\xi^2. \tag{17}$$

We demand the sum of the expressions in Eqs. (16) and (17) to vanish for any ξ. This requirement leads to the following three equations:

$$2(2b_2 - 12b_1 + 12b_0) + 2PL^2 = 0, \tag{18}$$

$$6(-12b_2 + 12b_1) - 12PL^2 = 0, \tag{19}$$

$$144b_2 + 12PL^2 = 0. \tag{20}$$

From the latter equation (20), we get the buckling load,

$$P = -12\frac{b_2}{L^2}, \tag{21}$$

which, remarkably, coincides with Eq. (10). Eqs. (18) and (19) lead to

$$b_0 = -b_2/6, \quad b_1 = -b_2. \tag{22}$$

Hence, we obtain the sought variation of the stiffness,

$$D(\xi) = \left(\tfrac{1}{6} + \xi - \xi^2\right)|b_2|, \tag{23}$$

which takes a positive value throughout the column's axis, $\xi \in [0; 1]$.

5. Column simply supported at one end and clamped at the other

The boundary conditions read

$$\begin{aligned} &w = 0, \quad D(\xi)\frac{d^2w}{d\xi^2} = 0, \quad \text{at } \xi = 0, \\ &w = 0, \quad \frac{dw}{d\xi} = 0, \quad \text{at } \xi = 1. \end{aligned} \tag{24}$$

The boundary conditions are satisfied by the following polynomial function:

$$\psi(\xi) = \xi - 3\xi^3 + 2\xi^4. \tag{25}$$

Substitution of this expression into the governing differential equation, in conjunction with postulated expression for the stiffness results in

$$\left[2(-18b_1 + 24b_0) + 6(-18b_2 + 24b_1)\xi + 288b_2\xi^2\right] + PL^2(-18\xi + 24\xi^2) = 0. \tag{26}$$

Since Eq. (26) is valid for any ξ, we get following three equations:

$$-18b_1 + 24b_0 = 0 \quad \text{for } \xi^0, \tag{27}$$

$$6(-18b_2 + 24b_1) - 18PL^2 = 0 \quad \text{for } \xi^1, \tag{28}$$

$$288b_2 + 24PL^2 = 0 \quad \text{for } \xi^2. \tag{29}$$

We arrive at three equations for four unknowns: b_0, b_1, b_2 and P. We choose one of the parameters to be arbitrary, namely b_2. Eq. (29) yields the same buckling load as in Eqs. (10) and (21):

$$P = -12\frac{b_2}{L^2}. \tag{30}$$

Eqs. (27)–(29) yield the following interrelation between the coefficients describing the stiffness variation:

$$b_0 = -\tfrac{9}{16}b_2, \quad b_1 = -\tfrac{3}{4}b_2. \tag{31}$$

Substituting into Eq. (3) results in the variation of the stiffness:

$$D(\xi) = \left(\tfrac{9}{16} + \tfrac{3}{4}\xi - \xi^2\right)|b_2|, \tag{32}$$

which is a positive function within the length of the column. The functions $D(\xi)$, for all three cases, are depicted in Figs. 1–3.

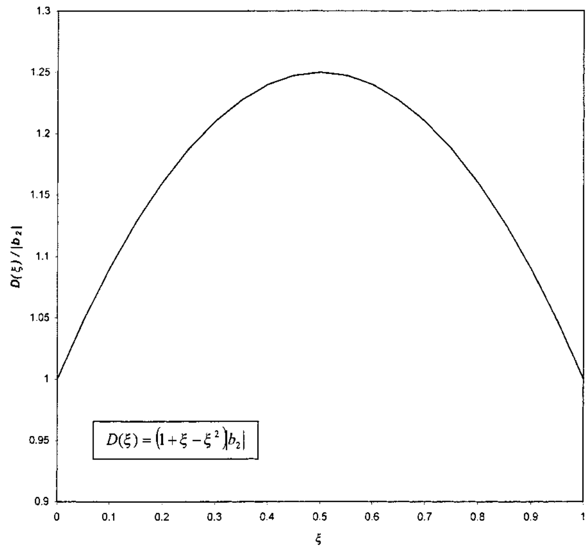

Fig. 1. Variation of $D(\xi)/|b_2|, \xi \in [0;1]$, for the column that is simply supported at both its ends.

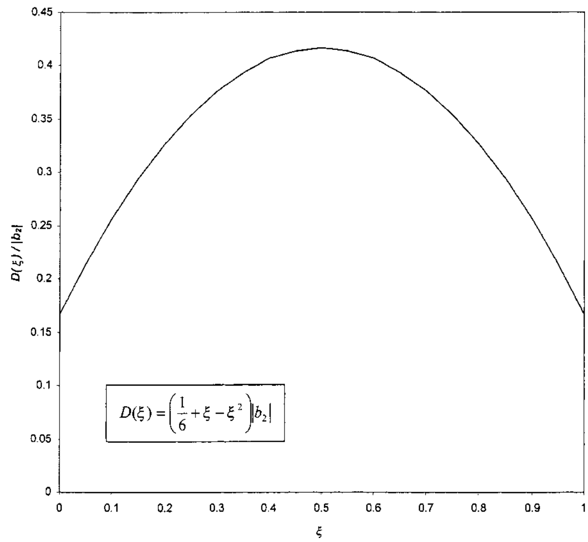

Fig. 2. Variation of $D(\xi)/|b_2|, \xi \in [0;1]$, for the column that is clamped at both its ends.

6. Discussion

The following conclusions appear to be relevant:

(1) Inverse buckling problems with postulated polynomial buckling modes, as given in Eq. (2), Eq. (15) or Eq. (25) for corresponding boundary conditions have closed-form solutions; namely, the variations of stiffness corresponding to the above mode shapes are given in Eqs. (13), (23) and (32), respectively.

(2) For three sets of boundary conditions, the fundamental buckling load is given by the same expression (Eqs. (10), (21) and (30)). This conclusion may appear to be a paradoxical one at first glance. To resolve it,

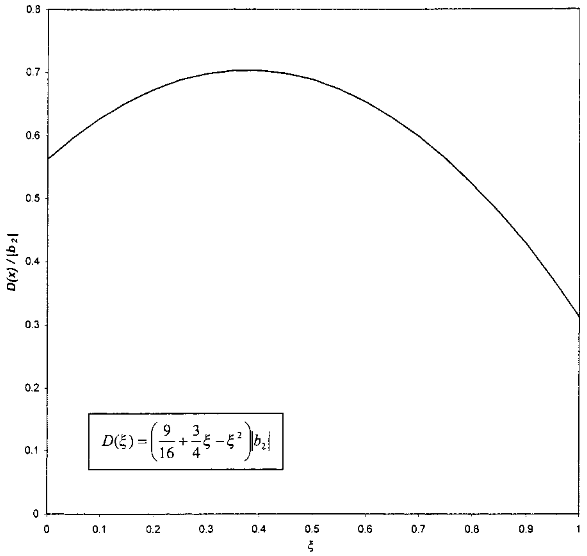

Fig. 3. Variation of $D(\xi)/|b_2|, \xi \in [0;1]$, for the column that is simply supported at one end and clamped at the other.

let us consider the corresponding case of homogeneous and uniform columns. The fundamental buckling loads are

$$P_{SS} = \frac{\pi^2 D_1}{L^2}, \qquad P_{CC} = \frac{4\pi^2 D_2}{L^2}, \qquad P_{SC} = \frac{4.493^2 D_3}{L^2} \approx \frac{2\pi^2 D_3}{L^2}, \tag{33}$$

where D_1 is the stiffness of the column that is simply supported at its both ends, D_2 corresponds to the column that is clamped at its both ends; D_3 is associated with the column that is simply supported at $x = 0$ and clamped $x = L$. These three columns possess the same cross-sections and have the same lengths. Now, if

$$D_1 = 4D_2 \approx 2D_3, \tag{34}$$

then all three columns have coalescing buckling loads. This suggests that columns with different stiffnesses, if they are under different boundary conditions, but have the same lengths and cross-sections, *may* share the same fundamental buckling load. The same phenomenon takes place in the case of our study. The columns with three different sets of boundary conditions share the same fundamental buckling load, and their moduli of elasticity are different.

Still, it appears to be intriguing that the search of the solution of inverse buckling problem in the class of polynomial functions lead to the coincidence of buckling loads for simply supported–simply supported, clamped–clamped and simply supported–clamped boundary conditions. Here, we reported the case when the fundamental buckling loads are shared by columns under different boundary conditions (see also related studies Gottlieb, 1989 and Gladwell and Morassi, 1995, for vibration problems).

(3) To compare the results for the buckling loads, let us calculate the average stiffness in each of three cases. Average stiffness is defined as

$$D_{av} = \int_0^1 D(\xi)\,d\xi. \tag{35}$$

Thus, for a simply supported column,

$$D_{av,SS} = \tfrac{7}{6}|b_2|. \tag{36}$$

For a clamped column,

$$D_{\mathrm{av,CC}} = \tfrac{1}{3}|b_2|. \tag{37}$$

For a column that is simply supported at one end and clamped at the other,

$$D_{\mathrm{av,SC}} = \tfrac{29}{48}|b_2|. \tag{38}$$

Thus, the buckling loads can be put in the following forms, by first expressing $|b_2|$ via D_{av} in Eqs. (36)–(38):

$$P_{\mathrm{SS}} = \frac{12|b_2|}{L^2} = \frac{72}{7}\frac{D_{\mathrm{av}}}{L^2}, \tag{39}$$

$$P_{\mathrm{CC}} = \frac{12|b_2|}{L^2} = 36\frac{D_{\mathrm{av}}}{L^2}, \tag{40}$$

$$P_{\mathrm{SC}} = \frac{12|b_2|}{L^2} = \frac{576}{29}\frac{D_{\mathrm{av}}}{L^2}. \tag{41}$$

If the average stiffnesses of these columns are chosen to be the same, then the buckling loads of the inhomogeneous columns are in the proportion

$$\tfrac{72}{7} : 36 : \tfrac{576}{29} \tag{42}$$

or 1:3.5:1.93, versus the corresponding proportion 1:4:($\approx$2) for the uniform columns.

(4) For uniform columns, the polynomial expressions of the buckling mode are usually utilized to facilitate the approximate solutions, via the Bubnov–Galerkin, Rayleigh, or Rayleigh–Ritz methods. For example, Chajes (1974) uses the function $w(x) = xL^3 - 3x^3L + 2x^4$, as a *comparison function* in the Bubnov–Galerkin method for the column that is simply supported at $x = 0$ and clamped at $x = L$. It is interesting that the same polynomial function (coincident with Eq. (25)) turned out to be an *exact* buckling mode of the *inhomogeneous* column.

Likewise, in his book of problems, Volmir (1984) poses a question of using the Rayleigh's method as well as the Bubnov–Galerkin method for the *approximate* estimation of the buckling load of an uniform column, simply supported at its both ends, by utilizing the trial function $w(x) = xL^3 - 2x^3L + x^4$. In our case, the same buckling mode (coincident with Eq. (2)) serves as an *exact* expression of the inhomogeneous column.

(5) A question arises on the generality of the proposed method. In order that the method be acceptable, it should lead to the positive buckling load, corresponding to the case of a compressive force, and modulus of elasticity that is a positive function. The formulation of some general conditions upon fulfillment of which the problem is amenable to a solution is of interest.

(6) The buckling loads turn out to depend upon only a single coefficient b_2, once the function of modulus variation is obtained. By suitable choice of b_2, the buckling load can be made arbitrarily large. This conclusion is valid within the context of elastic buckling that was pre-supposed in this study.

(7) The solution of the buckling problems of uniform columns leads to irrational values of the buckling loads in known cases; for example, the buckling load of the simply supported *uniform* column is written in terms of π^2 (Eq. (33)). So is the buckling load, found by Euler (1757) in the case of the variable stiffness column, where the buckling load is $\pi^2(D_0/L^2)a^2(a+b)^2$ for the column with the stiffness $D(\xi) = D_0(a+b\xi)^4$ (see also works by Dinnik, 1932, 1955). Here, in the case of *inhomogeneous* and/or *non-uniform* column, the solution turns out to be in terms of rational numbers. Other solutions of this kind (namely by Duncan, 1937) are reported by Elishakoff and Rollot (1999). It must be stressed that this conclusion is true for special cases herein discussed, and it may be inapplicable for other cases.

(8) In this study, we assume $D(\xi)$ to be the polynomial, which fits the solution. The natural question arises: Can we use other functions, such as *sin* or *cos*, or some other special functions instead of polynomials? It appears that this question ought to be investigated in the future studies.

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References

Chajes, A., 1974. Principles of Structural Stability Theory. Prentice Hall, Englewood Cliffs, p. 106.

Dinnik, A.N., 1932. Design of column of varying cross-section. Transactions of the ASME, Applied Mechanics.

Dinnik, A.N., 1955. Prodol'nyi Izgib. Kruchenie (Bending in Presence of Axial Forces. Torsion). Academy of USSR Publishing, pp. 96–130 (in Russian).

Duncan, W.J., 1937. Galerkin's method in mechanics and differential equations. ARC, Reports and Memoranda, No. 1738.

Elishakoff, I., Rollot, O., 1999. New closed-form solutions for buckling of a variable stiffness column by mathematica. Journal of Sound and Vibration 224, 172–182.

Euler, L., 1757. Sur la force de collones, Memoires de l'Académie Royale des Sciences et Belles Lettres, Berlin, vol. 13, pp. 252–282 [see also in: Leonhardi Euleri Opera Omnia, Bern, vol. 17, Part 2, pp. 89–118, 1982 (in Latin)].

Glawdwell, G.M.L., Morassi, A., 1995. On isosepectral rods, beams and strings. Inverse Problems 11, 533–554.

Gottlieb, H.P.W., 1989. On standard eigenvalues of variable-coefficient heat and rod equations. Journal of Applied Mechanics 56, 146–148.

Timoshenko, S.P., Gere, J. M., 1961. Theory of Elastic Stability. McGrawHill, New York.

Volmir, A.S. (Ed.), 1984. Collection of Problems in Strength of Materials. Nauka Publishers, Moscow, p. 30 (problem 7.34, in Russian).